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Many observers can simultaneously measure different parts of an environment of a quantum system in order to find out its state. To study this problem we generalize the formalism of conditional master equations to the multiple observer case. To settle some issues of principle which arise in this context (as the state of the system and of the environment are ultimately correlated), we consider an example of a system qubit interacting through controlled nots (CNOTs) with environmental qubits. The state of the system is the easiest to find out for observers who measure in a basis of the environment which is most correlated with the pointer basis of the system. In this case the observers agree the most. Furthermore, the more predictable the pointers are, the easier it is to find the state of the system, and the better is the agreement between different observers.

Pointer states are the states of a system which get entangled the least with the environment. They are therefore the most predictable and, hence, the most classical states of the system [1–3]. In our recent Letter [4] we considered an observer performing continuous quantum measurements on an environment of a system in order to monitor and predict its state [5,6]. We found that under reasonable assumptions pointer states of the system do not depend on the basis selected by the observer to carry out measurements on the environment. We also found evidence that measurements in a basis of the environment which is most strongly correlated with the system are most efficient in yielding information about its state.

In this Letter we consider several observers monitoring different parts of the environment to extract information about the system. We shall show that, again, each observer gains most information from measurements in a basis which is most strongly correlated with the pointer states. However, in the presence of multiple observers new questions arise about correlations between the state different observers ascribe to the system. We find that when all observers measure their environments in a basis correlated to the pointer states, then the indications of their appartuses are very strongly correlated, as might have been expected for measurements of very classical states. On the other hand, when observers measure in a basis poorly correlated to the pointer states or when the preferred pointer states are not very classical, then it is possible that apparatuses disagree for large fraction of time (see also [7] for an information theoretic discussion of related issues).

Many-body entanglement occurs in course of decoherence when several subsystems of the environment get entangled with the system. For instance, a one-qubit system and, say, two one-qubit environments can find themselves in a GHZ-like state

$$\frac{1}{\sqrt{2}}\left(|1\rangle_{\rm S}|1\rangle_{\rm E_1}|1\rangle_{\rm E_2} \;+\; |0\rangle_{\rm S}|0\rangle_{\rm E_1}|0\rangle_{\rm E_2}\right) \quad . \tag{1}$$

We shall focus on this ideal case, as it allows us to illustrate interesting issues of fundamental importance that arise in the case of multiple observers.

The reduced density matrix of the system is mixed, $\rho = (|1\rangle\langle 1| + |0\rangle\langle 0|)/2$. Imagine that environments E_1 and E_2 are measured by different observers 1 and 2 who know beforehand that the total state is (1). Observer 1 measures the state of E_1 in the $\{|1\rangle, |0\rangle\}$ basis. If his measurement result is $|1\rangle(|0\rangle)$, then he discovers that the system (and, by the way, the other environment) are in the state $|1\rangle(|0\rangle)$. If his measurement is followed by the measurement of observer 2, then observer 2 will also find both his environment and the system in the $|1\rangle(|0\rangle)$ state. The $\{|1\rangle, |0\rangle\}$ basis is a good choice in the sense that each observer alone can find out about the system state.

Suppose that the observers want to find out about the state of the system in another basis, say the Hadamard transformed basis $|\pm\rangle = (|1\rangle \pm |0\rangle)/\sqrt{2}$. Suppose that observer 1 made a measurement in this basis and that his outcome is "+". His measurement projects GHZ state (1) onto $|1\rangle_{\rm S}|+\rangle_{\rm E_1}|1\rangle_{\rm E_2} \;+\; |0\rangle_{\rm S}|+\rangle_{\rm E_1}|0\rangle_{\rm E_2}$. The reduced density matrix of the system is not affected at all: it remains in the initial mixed state as before the measurement. The single observer can find nothing about the state of the system when he measures in the "wrong" $\{|+\rangle, |-\rangle\}$ basis. Let observer 2 step in and make his measurement. If he measures in the "good" basis, then he gets full information about the system. If, on the contrary, he measures in the "wrong" basis, alone he will not be able to ascertain the state of the system. Suppose that the result of his measurement is, say, "−". Then the previous state is further projected on $|-\rangle_{\rm S}|+\rangle_{\rm E_1}|-\rangle_{\rm E_2}$, and the system is in the pure "−" state, $\rho = |-\rangle\langle -|$. Having measured E_1 and E_2 in the $\{|+\rangle, |-\rangle\}$ basis, each observer alone is ignorant of the state of the system. However, correlations between their measurement results contain full information about the system. If the outcomes of the two observers are “++” or “−−” then the system state is “+”, but if the outcomes are “+−” or “−+” then the system state is “−”.

In the GHZ example above the state (1) was known to both observers beforehand. Given that knowledge, and after a fortuitous choice of the observables, they could draw unambiguous conclusions about the state of the system after just one projection. In practice correlations between an unknown state of the system and different parts of the environment arise as a result of interaction. Let us consider a toy example that illustrates such a scenario. Let the system be a single qubit $\mathcal{S}$ with zero self-Hamiltonian. It is initially prepared in a state

$$\rho^{t_0} = \sum_{a,b=0,1} \rho_{ab}^{t_0} \, |a\rangle\langle b| \ . \tag{2}$$

Let the environment of such a qubit be an ensemble of pairs of qubits, all initially prepared in state $|0\rangle$ ($|1\rangle$ and $|0\rangle$ are eigenstates of σ_z with eigenvalues $+1$ and -1 respectively). We want to find out the state of the system from measurements on the environment. The environmental qubits entangle with the system as shown in Fig.1. At the time t_1 the first pair of environmental qubits denoted by $(1, t_1)$ and $(2, t_1)$ is put in contact with the system. The system acts as a control on the environmental qubits, perfoming a C-NOT operation on both of them, so that the system qubit and the environmental pair of qubits get fully entangled. After completion of these operations the first pair is decoupled from the system. At the time t_2 a second pair is entangled with the system in a similar way. After n such double CNOT operations the total density matrix becomes

$$\begin{aligned} \rho_{\mathcal{S}+\mathcal{E}}^{t_n} = \sum_{a,b=0,1} & \rho_{ab}^{t_0} \, |a\rangle\langle b| \, \otimes \, |a\rangle\langle b|_{(1,t_1)} \, \otimes \, |a\rangle\langle b|_{(2,t_1)} \\ & \otimes \ldots \otimes \, |a\rangle\langle b|_{(1,t_n)} \, \otimes |a\rangle\langle b|_{(2,t_n)} \ . \end{aligned} \tag{3}$$

The reduced density matrix of the system after n steps, ρ^{t_n}, can be obtained from $\rho_{\mathcal{S}+\mathcal{E}}^{t_n}$ by tracing over the environment, $\rho^{t_n} = \mathrm{Tr}_{\mathcal{E}} \, \rho_{\mathcal{S}+\mathcal{E}}^{t_n} = \rho_{00}^{t_0} \, |0\rangle\langle 0| + \rho_{11}^{t_0} \, |1\rangle\langle 1|$ for any $n > 0$. It becomes diagonal in the $\{|0\rangle, |1\rangle\}$ basis already after the entanglement with the first pair. In the steps that follow ρ^{t_n} does not change any more. This description can be encapsulated in the following difference equation for the matrix elements

$$\rho_{ab}^{t_n} \ = \ \delta_{ab} \, \rho_{ab}^{t_{n-1}} \ . \tag{4}$$

This is a markovian master equation, as the next state of the system depends only on its immediate predecesor (and not on the history). Since it was obtained by tracing out the environment, it is an “unconditional” master equation (UME), in the sense that all the information about the environment is ignored. If the system is initially in one of the states $|0\rangle$ or $|1\rangle$, then the state of the system does not get entangled with the environment and it does not lose any purity. In other words, the states $|0\rangle$ and $|1\rangle$ are perfect pointer states.

Suppose that the information about the state of $\mathcal{E}$ is not ignored. Let there be two observers labelled $\alpha = 1, 2$ who measure the operators $\hat{\sigma}_\alpha = x_\alpha \sigma_x + y_\alpha \sigma_y + z_\alpha \sigma_z$, where $x_\alpha^2 + y_\alpha^2 + z_\alpha^2 = 1$, on the environmental qubits (α, t_n). We shall denote the state of the environmental qubit α after the measurement as $|N_\alpha^{t_n}\rangle$, where $N_\alpha^{t_n}$ (which can be either $+1$ or -1) is the result of the measurement. Since the σ_z basis is the one correlated with the pointer states, we expect that observers measuring in that basis will most efficiently gain information and agree the most about the state of the system [4].

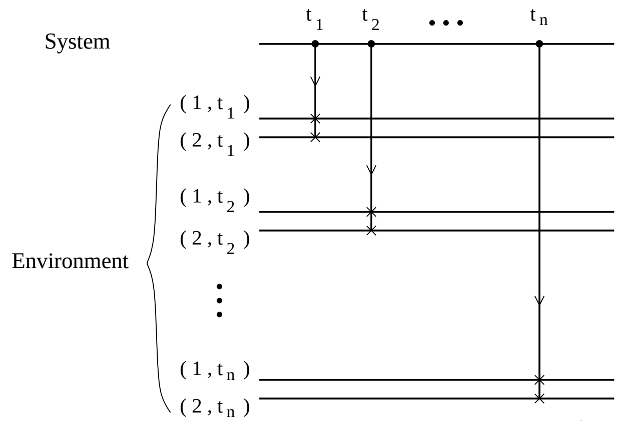

FIG. 1. C-NOT circuit for the toy model. Each pair (α, t_n) of qubits of the environment (target qubits) interacts only once with the system (control qubit). We recall that the logical operation C-NOT flips the state of the target if the state of the control is 1, and does nothing otherwise.

To make the measurement on an environmental qubit the observer fully entangles it with his measuring apparatus (in effect, using the environmental qubit as a control to perform CNOT in an eigenbasis of $\hat{\sigma}_\alpha$) and then the state of the memory decoheres in its pointer basis. The reduced density matrix of the memory qubit becomes diagonal in the measurement basis. Measurements on environmental qubits can be carried out at any time after they have interacted with the system. Their results affect the observers' knowledge about the state of the system at time t_n when the n-th pair of qubits got entangled with the system. This knowledge is expressed by the reduced density matrix of the system ρ^{t_n}. From the point of view of ρ^{t_n} this whole arbitrarily delayed measurement process can be described by the projections of the full $\rho_{\mathcal{S}+\mathcal{E}}^{t_n}$ in the measurement basis of the n-th pair of qubits.

After $n-1$ steps followed by $2(n-1)$ measurements with the outcomes $\{N_\alpha^{t_n}\}$ in the measurement basis of observers $\alpha = 1, 2$, the density matrix of the system and the environment, conditioned on given set of measurement records, is

$$\begin{aligned} \rho_{\mathcal{S}+\mathcal{E}}^{t_n} &= |N_1^{t_1}\rangle\langle N_1^{t_1}| \, \otimes \ldots \otimes \, |N_2^{t_{n-1}}\rangle\langle N_2^{t_{n-1}}| \\ \otimes & \left(\sum_{a,b=1,0} \rho_{ab}^{t_{n-1}} \, |a\rangle\langle b| \, \otimes \, |a\rangle\langle b|_{(1,t_n)} \, \otimes \, |a\rangle\langle b|_{(2,t_n)} \right) , \end{aligned} \tag{5}$$

where $\rho^{t_{n-1}}$ is the reduced density matrix of the system after $n-1$ steps conditioned on the measurement results at the times $t_1, \ldots, t_{n-1}$. The unnormalized conditional reduced density matrix of the system becomes

$$\begin{aligned}
\tilde{\rho}^{t_n} &= \mathrm{Tr}_{\mathcal{E}-\mathcal{E}_n} \langle N_1^{t_n}| \langle N_2^{t_n}| \, \rho_{\mathcal{S}+\mathcal{E}}^{t_n} \, |N_1^{t_n}\rangle \, |N_2^{t_n}\rangle \\
&= \frac{1}{4} \, \rho_{11}^{t_{n-1}} \, |1\rangle\langle 1| \, (1 + z_1 N_1^{t_n})(1 + z_2 N_2^{t_n}) \\
&+ \frac{1}{4} \, \rho_{00}^{t_{n-1}} \, |0\rangle\langle 0| \, (1 - z_1 N_1^{t_n})(1 - z_2 N_2^{t_n}) \\
&+ \frac{1}{4} \, N_1^{t_n} \, N_2^{t_n} \, \rho_{10}^{t_{n-1}} \, |1\rangle\langle 0| \, \sqrt{1 - z_1^2}\sqrt{1 - z_2^2} \\
&+ \frac{1}{4} \, N_1^{t_n} \, N_2^{t_n} \, \rho_{01}^{t_{n-1}} \, |0\rangle\langle 1| \, \sqrt{1 - z_1^2}\sqrt{1 - z_2^2} \quad . \qquad (6)
\end{aligned}$$

The probability to get a given outcome $(N_2^{t_n}, N_2^{t_n})$ and the normalized reduced density matrix of the system conditioned on this outcome are

$$\begin{aligned}
P(N_1^{t_n}, N_2^{t_n} | \rho^{t_{n-1}}) &= \mathrm{Tr} \, \tilde{\rho}^{t_n} \qquad (7) \\
&= \frac{1}{4} \, \rho_{11}^{t_{n-1}} \, (1 + z_1 N_1^{t_n})(1 + z_2 N_2^{t_n}) \\
&+ \frac{1}{4} \, \rho_{00}^{t_{n-1}} \, (1 - z_1 N_1^{t_n})(1 - z_2 N_2^{t_n}); \\
\rho^{t_n} &= \frac{\tilde{\rho}^{t_n}}{P(N_1^{t_n}, N_2^{t_n} | \rho^{t_{n-1}})} \; . \qquad (8)
\end{aligned}$$

The above equation is a multiple observer conditional master equation (MOCME). It describes the evolution of the knowledge about the state of the system ρ^{t_n} of a "supervisor" who has access to the measurement records $N_\alpha^{t_n}$ of all the observers. The average of the conditional ρ^{t_n} over different outcomes $N_\alpha^{t_n}$ weighted by their probability distribution $P(N_1^{t_n}, N_2^{t_n}|\rho^{t_{n-1}})$ gives the unconditional master equation (4).

Suppose that both observers measure in the σ_z basis. The probability distribution $P(N_1^{t_n}, N_2^{t_n}|\rho^{t_{n-1}}) = \delta_{N_1^{t_n}, N_2^{t_n}} \, (\rho_{11}^{t_{n-1}} \, \delta_{N_1^{t_n},+1} \; + \; \rho_{00}^{t_{n-1}} \, \delta_{N_1^{t_n},-1})$ implies that the results of the two observers are completely correlated, $N_1^{t_n} = N_2^{t_n}$. The full correlation follows from the entanglement between the system and the environmental qubits. The result $N_1^{t_n} = N_2^{t_n} = +1(-1)$ obtained with the probability $\rho_{11}^{t_{n-1}}(\rho_{00}^{t_{n-1}})$ gives a conditional $\rho^{t_n} = |1\rangle\langle 1|(\rho^{t_n} = |0\rangle\langle 0|)$. In the σ_z-basis it is enough to measure just one environmental qubit to gain full knowledge about the state of the system (and purify ρ^{t_n}). The fully correlated observers always agree that this state is $|1\rangle$ or $|0\rangle$. We note again that a weighted average over the results $N_\alpha^{t_n}$ gives the unconditional $\rho^{t_n} = \rho_{11}^{t_{n-1}} \, |1\rangle\langle 1| \; + \; \rho_{00}^{t_{n-1}} \, |0\rangle\langle 0|$ in agreement with the UME (4).

It is not so easy to find the state of the system when the observers measure in a basis which is less well correlated to the pointer states. To illustrate this we take for definiteness $z_1 = z_2 = \epsilon$ with $0 \le \epsilon \ll 1$. We will follow the evolution of the observers' knowledge about the state of the system with the help of the polarization $A^{t_n} = \rho_{11}^{t_n} - \rho_{00}^{t_n}$ they infer from their measurements. Full knowledge of a predictable pointer state, a pure $|0\rangle$ or $|1\rangle$ state of the system, corresponds to $A^{t_n} = \pm 1$. We use the MOCME (8) and expand to leading order in ϵ to derive a stochastic master equation for A^{t_n},

$$\begin{aligned}
\frac{A^{t_n} - A^{t_{n-1}}}{\epsilon} &= (N_1^{t_n} + N_2^{t_n})[1 - (A^{t_{n-1}})^2] \; , \\
P(N_1^{t_n}, N_2^{t_n}|A^{t_{n-1}}) &= \frac{1}{4} \, [\, 1 \; + \; \epsilon(N_1^{t_n} + N_2^{t_n}) A^{t_{n-1}} \,] \, . \qquad (9)
\end{aligned}$$

Measurements of σ_z yield $A^{t_n} = \pm 1$, that are fixed points of this equation. For $\epsilon = 0$ (e.g. σ_x-measurement) the polarization A^{t_n} does not change at all: if we start from $-1 < A^{t_0} < +1$, then using such $\epsilon = 0$ measurements observers will never find out whether A^{t_n} is $+1$ or -1. The outcomes of their measurements do not depend on the state of the system. For $0 < \epsilon \ll 1$ patient observers will find out the state of the system if they measure $\propto 1/\epsilon^2$ environment. The polarization A^{t_n} makes a random walk. When A^{t_n} walks into the area $A > 0$, then in the next measurement the sum $N_1^{t_n} + N_2^{t_n}$ will more likely come out positive than negative and it will probably drive A^{t_n} to be even more positive. Eventually, after $\approx 1/\epsilon^2$ environmental qubits get entangled with the system, A^{t_n} will settle at $+1$ or at -1. The closer is the measurement basis correlated with the pointer states, the faster it is to find the state of the system.

So far we have described the evolution of ρ^{t_n} as if we knew the records of both observers. This is rarely the case. Suppose that observer 1 knows only his own records $N_1^{t_n}$. What density matrix $\rho_1^{t_n}$ represents his knowledge about the state of the system? Since he does not know $N_2^{t_n}$ the best he can do is to treat the other observer as if he were an environment, i.e., assume for $N_2^{t_n}$ a probability distribution like in Eq.(7) and average the $\tilde{\rho}^{t_n}$ in Eq.(6) over $N_2^{t_n}$ with this distribution. The weighted average is $\tilde{\rho}_1^{t_n} = (\rho_{11}^{t_{n-1}} \, |1\rangle\langle 1| \, (1 + z_1 N_1^{t_n}) + \rho_{00}^{t_{n-1}} \, |0\rangle\langle 0| \, (1 - z_1 N_1^{t_n}))/2$. The right hand side (RHS) of this equation still depends on the multiple observer $\rho^{t_{n-1}}$ but the observer 1 does not know $\rho^{t_{n-1}}$ because he does not know any earlier records of observer 2. In this situation the best he can do is to take an average of the RHS over the earlier records of observer 2: $N_2^{t_{n-1}}, N_2^{t_{n-2}}, \ldots, N_2^{t_1}$. By definition, this average replaces ρ^{t_n} on the RHS by the single observer density matrix $\rho_1^{t_n}$. After normalisation, so that $\mathrm{Tr}\rho_1^{t_n} = 1$, we obtain a single observer conditional master equation (SOCME) for the observer $\alpha = 1$, conditioned only on his own records $N_\alpha^{t_n}$,

$$\rho_\alpha^{t_n} = \frac{\rho_{\alpha,11}^{t_{n-1}} \, |1\rangle\langle 1| \, (1 + z_\alpha N_\alpha^{t_n}) \; + \; \rho_{\alpha,00}^{t_{n-1}} \, |0\rangle\langle 0| \, (1 - z_\alpha N_\alpha^{t_n})}{\rho_{\alpha,11}^{t_{n-1}} \, (1 + z_\alpha N_\alpha^{t_n}) \; + \; \rho_{\alpha,00}^{t_{n-1}} \, (1 - z_\alpha N_\alpha^{t_n})} \; . \qquad (10)$$

The SOCME gives us a tool to check if and how fast do the observers 1 and 2 reach agreement about the state of the system. To this end we define "single observer" polarizations $A_\alpha^{t_n} = \rho_{\alpha,11}^{t_n} - \rho_{\alpha,00}^{t_n}$. We use Eq.(10) and an expansion to leading order in ϵ to derive a stochastic equation for $A_\alpha^{t_n}$ conditioned on $N_\alpha^{t_n}$,

$$\frac{A_\alpha^{t_n} - A_\alpha^{t_{n-1}}}{\epsilon} = N_\alpha^{t_n} \left[1 - (A_\alpha^{t_{n-1}})^2\right] . \qquad (11)$$

If the supervisor's polarization A^{t_n} finally settles at ± 1, then the probability distribution in Eq.(7) will prefer positive values for both $N_\alpha^{t_n}$'s and both $A_\alpha^{t_n}$'s will follow A^{t_n} to ± 1 (see Fig. 2). If the observers finally meet and compare their results, they will fully agree. In order to get this agreement it is needed to entangle with the system an amount $\propto 1/\epsilon^2$ of pairs of environmental qubits, which is the smaller the better are the measured states of the environment correlated to the pointer states.

We derived the SOCME by averaging over the unknown records of the other observer. Note that the probability distribution in (8) is linear in $\rho^{t_{n-1}}$ and can be easily averaged over the earlier records of the observer 2 at the earlier times $t_{n-1}, t_{n-2}, \ldots, t_0$. As a result the ρ^{t_n} in the distribution (8) averages to $\rho_1^{t_{n-1}}$. Such a partially averaged distribution can be traced over $N_2^{t_n}$ to give a distribution for a single observer $\alpha = 1$,

$$P_\alpha(N_\alpha^{t_n}|\rho_\alpha^{t_{n-1}}) = \frac{1}{2}\, \rho_{\alpha,11}^{t_{n-1}}\, (1 + z_\alpha N_\alpha^{t_n}) + \frac{1}{2}\, \rho_{\alpha,00}^{t_{n-1}}\, (1 - z_\alpha N_\alpha^{t_n}) \ . \qquad (12)$$

Eqs. (10,12) can be regarded as a single observer stochastic generator for the string of records $\{N_\alpha^{t_n}\}$. The multiple observer conditonal master equation (8) can also be regarded as a "multiple observer" stochastic generator for the two strings of records $\{N_1^{t_n}, N_2^{t_n}\}$. From our derivation it is clear that if we are given just one string, say, $\{N_1^{t_n}\}$, then we will not be able to find out if the string comes from the multiple observer or from the single observer generator. If records were not independent, causality would be in trouble. One could send the entangled qubit 2 to a distant galaxy where the observer 2 would make his measurements. By choosing to measure or not to measure, or by changing the measurement basis, 2 could affect records $N_1^{t_n}$ of the other observer and could signal with superluminal velocity or backwards in time. While causality is a general and fundamental requirement, record independence is assured by the nature of our environment. Each qubit pair is put in contact with the system only once. Once they decouple they can no longer perturb the system state. On the other hand, if they did not decouple, then in general it would be possible to find out what observer 2 is doing without violating causality.

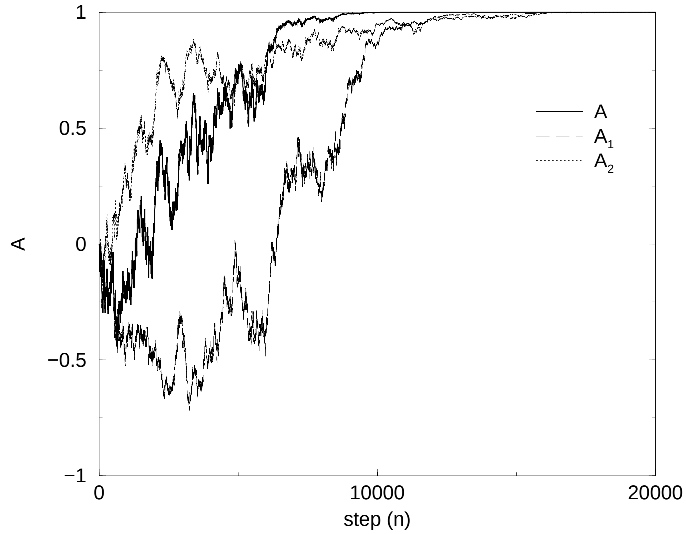

FIG. 2. A single realization of the stochastic trajectories for the polarizations A^{t_n}, $A_1^{t_n}$ and $A_2^{t_n}$. The initial condition for all of them is null polarization. In this figure $\epsilon = 10^{-2}$.

In conclusion, we have shown that when several observers perform measurements on the environment of a system, they agree most about the state of the system if their measurement basis are correlated with the pointer states. For any other measurement basis their gain of information is less efficient, and they can even gain no information at all if they choose a "wrong" measurement basis. These results can be generalized to the more realistic (but also more cumbersome) case of continuous quantum measurement [8].

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[1] W.H. Zurek, Phys.Rev.**D24**, 1516 (1981); *ibid.* **26**, 1862 (1982).
[2] W.H. Zurek, Prog.Theor.Phys. **81**, 281 (1993).
[3] W.H. Zurek, S. Habib and J.P. Paz, Phys. Rev. Lett. **70**, 1187 (1993).
[4] D.A.R. Dalvit, J. Dziarmaga and W.H. Zurek, Phys. Rev. Lett. **86**, 373 (2001).
[5] H.J. Carmichael, *An Open Systems Approach to Quantum Optics* (Springer, Berlin, 1993).
[6] H.M. Wiseman and G.J. Milburn, Phys. Rev. Lett. **70**, 548 (1993); H.M. Wiseman, Ph.D. thesis, University of Queensland, 1994.
[7] W.H. Zurek, Annalen der Physik **9**, 855 (2000); Rev. Mod. Phys., submitted.
[8] J. Dziarmaga, D.A.R. Dalvit and W.H. Zurek, in preparation.